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Five-Dimensional Null-Cone Structure of Big Bang Singularity

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The Friedmann model Φ of positive space curvature, vanishing pressure and cosmological constant when isometrically imbedded as a hypersurface in five-dimensional Minkowski space M^5 is globally rigid: if $\underline{F}(\Phi)$ and $\underline{F}'(\Phi)$ are isometric imbeddings in M^5 there is a motion π of M^5 such that $\underline{F}' = \pi \circ \underline{F}$. The big bang singularity is the vertex of a null half-cone in M^5. Global rigidity leads to an invariant characterization of the singularity. The structure of matter at the singularity is governed by the de Sitter group.

1. INTRODUCTION

Many of the Friedmann models of the cosmos lead to a singularity (Robertson, 1933): for a cosmic time $t \to 0$ the density of matter becomes infinite. This leads—via the Einstein equations for gravitation—to an intrinsic singularity of the space-time metric known as the big bang or the creation of the universe.

It can be shown that models with positive curvature which include Φ and the radiation cosmos (Schucking and Wang, to be published) can be uniquely and invariantly extended to singular space-times which are compact topological—but not differentiable—manifolds that include the big bang and big crunch singularities.

We discuss in the following—as a representative case—the geometry of the model Φ.

We shall first show that the Riemann tensor of Φ defines uniquely (up to one overall sign on Φ) a symmetric tensor h_{ij} as a sort of square root (Theorem 1). h_{ij} has all the properties of a second fundamental form which

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describes the variation of a normal vector to the manifold Φ if it is isometrically imbedded into a five-dimensional Minkowski space M^5. The two signs for h_{ij} correspond to the choice between inner and outer normal. With an intrinsically determined second fundamental form the imbedding of Φ into M^5 becomes thus locally rigid.

We claim then (Theorem 2) that this imbedding $\underline{F}(\Phi)$ into M^5 given by (Robertson, 1933) is also globally rigid. By introducing the scaffolding of M^5 for the Friedmann manifold we have created a convenient artifice for studying the properties of Φ especially near the singularities. The global rigidity assures us that all properties of $\underline{F}(\Phi)$ invariant under rigid motion of M^5 are intrinsic to Φ.

We find that Φ can be approximated near the singularities by null half-cones in M^5. Thus a picture of simplicity and beauty emerges for the singularities. Other approaches to the singularities will be discussed elsewhere (Lauro and Schucking, to be published).

2. THE FRIEDMANN MANIFOLD Φ

The manifold Φ is defined as a diffeomorph of $L \times S^3$, the direct product of the manifolds L and S^3. L is the real line segment $|\tau| < \pi$ and S^3 is the three-dimensional unit sphere. An event p of Φ is given by

$$p = \{\tau, X\}, X^+X = \mathrm{I}, \det X = 1$$

where X is an element of SU_2, i.e., a point of S^3, I the 2×2 unit matrix, and the "+" superscript is the Hermitian conjugate of a matrix.

On Φ four real differential one-forms

$$(\omega^i) = (\omega^0, \omega^\alpha) = (\omega^0, \boldsymbol{\omega})$$

are given by

$$\omega^0 = R(\tau)\, d\tau, \qquad \omega^\alpha = R(\tau)\lambda^\alpha$$

$$i\boldsymbol{\lambda}\cdot\boldsymbol{\sigma} = i\lambda^\alpha\sigma_\alpha = X^+\, dX = -(dX^+)X, \qquad \mathrm{Tr}\, X^+\, dX = 0$$

Lower case Latin indices run from 0 to 3, lower case Greek ones from 1 to 3. R as function of τ is

$$R(\tau) = a(1 + \cos\tau), \qquad a = \text{const} > 0$$

The σ_α are the three Pauli matrices. The λ^α are three real linearly independent left-invariant differential one-forms on SU_2.

A Lorentz metric is defined on Φ by

$$ds^2 = -(\omega^0)^2 + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 = \eta_{ij}\omega^i\omega^j$$
$$= R^2(\tau)(-d\tau^2 + \boldsymbol{\lambda}\cdot\boldsymbol{\lambda}) = -dt^2 + R^2 \operatorname{Tr} dX\, dX^+/2$$

τ is the regularizing parameter also known as subjective time counted positively from the moment of maximal spatial extension of Φ. It is related to the cosmic time t, the proper time measured by an observer at constant X counted from the big bang, by

$$t = a(\tau + \sin\tau + \pi)$$

R as a function of t is a cycloid.

The manifold Φ is analytic, simply connected, orientable, time orientable, and parallelizable.

The particular function R chosen turns Φ into a solution of the Einstein field equations with zero cosmological term for a perfect fluid with vanishing pressure and a four-velocity one-form of matter given by $-\omega^0$.

From the differential forms ω^i on Φ one obtains the connection one-forms (Misner, Thorne, and Wheeler, 1973)

$$\omega_{kj} = -\omega_{jk} = \eta_{ki}\omega^{i}_{\cdot j}$$

as the unique solutions of the equations

$$0 = d\omega^i + \omega^{i}_{\cdot j}\Lambda\omega^j$$

This gives with

$$i\, d\boldsymbol{\lambda}\cdot\boldsymbol{\sigma} = dX^+\Lambda\, dX \qquad \text{or} \qquad d\lambda_\alpha = \varepsilon_{\alpha\beta\gamma}\lambda^\beta\Lambda\lambda^\gamma$$

the connection one-forms

$$\omega_{\alpha 0} = \dot{R}R^{-2}\omega_\alpha, \qquad \omega_{\alpha\beta} = R^{-1}\varepsilon_{\alpha\beta\gamma}\omega^\gamma$$

where a dot on R denotes differentiation with respect to τ.

The curvature form Ω_{ik} is defined by

$$\Omega_{ik} = d\omega_{ik} + \omega_{ij}\Lambda\omega^{j}_{\cdot k} = R_{iklm}\omega^l\Lambda\omega^m/2$$

which leads for Φ to

$$\Omega_{0\alpha} = -aR^{-3}\omega_0\Lambda\omega_\alpha, \qquad \Omega_{\alpha\beta} = 2aR^{-3}\omega_\alpha\Lambda\omega_\beta$$

With one-forms χ_i defined by

$$\chi_0 = (a/2R^3)^{1/2}\omega_0, \qquad \chi_\alpha = -(2a/R^3)^{1/2}\omega_\alpha$$

one obtains

$$\Omega_{ij} = \chi_i \Lambda \chi_j, \qquad \chi_i \Lambda \omega^i = 0, \qquad \chi_0 \Lambda \chi_1 \Lambda \chi_2 \Lambda \chi_3 \neq 0$$

We prove now the following:

Theorem 1 (local rigidity of Φ). The differential forms χ_i defined as solutions of the Gauss equations

$$\Omega_{ij} = \chi_i \Lambda \chi_j$$

on Φ are defined up to one overall sign.

Proof of Theorem 1. Let

$$\alpha_i = b^j_i \chi_j, \qquad \alpha_i \Lambda \alpha_j = \chi_i \Lambda \chi_j$$

Then—with no summation over the index "i" indicated by underlining—

$$0 = \alpha_{\underline{i}} \Lambda \alpha_{\underline{i}} \Lambda \alpha_k = b^j_{\underline{i}} \chi_j \Lambda \chi_{\underline{i}} \Lambda \chi_k$$

Thus,

$$b^j_i = 0, \qquad i \neq j$$

since for all pairs of indices $i \neq j$ an index k different from i and j exists and the forms are linearly independent. Therefore,

$$\alpha_{\underline{i}} = b_{\underline{i}} \chi_{\underline{i}}$$

with proportionality factors b_i. It follows for $i \neq j$, $k \neq j$, $b_i b_j = b_j b_k = 1 \rightarrow b_i = b_k = b \rightarrow b^2 = 1 \rightarrow b = \pm 1$ and thus finally

$$\alpha_i = \chi_i \qquad \text{or} \qquad \alpha_i = -\chi_i$$

The sign b is globally defined on Φ since Φ is simply connected.

We can thus construct on Φ globally in an invariant manner by means of the curvature form a tensor h_{ij} by putting

$$\chi_i = h_{ij} \omega^j$$

This tensor is symmetric because of

$$\chi_i \Lambda \omega^i = 0$$

and unique—up to an overall sign. The tensor can be used to elucidate the structures of the big bang and big crunch singularities for Φ. For this purpose one can interpret h_{ij} as the coefficients of a second fundamental form II defined by

$$II = h_{ij} \omega^i \omega^j$$

provided that the equations of Codazzi and Mainardi

$$d\chi_j + \chi_i \Lambda \omega^{i}_{\cdot j} = 0$$

are fulfilled. This is indeed the case.

3. THE ROBERTSON IMBEDDING OF Φ

We describe five-dimensional Minkowski space M^5 by means of five coordinates y^A where capital Latin indices, like "A," run from 0 to 4. The metric of M^5 is given as

$$\begin{aligned} ds^2 &= -(dy^0)^2 + (dy^1)^2 + (dy^2)^2 + (dy^3)^2 + (dy^4)^2 \\ &= \eta_{AB}\, dy^A\, dy^B = d\underline{y} \cdot d\underline{y} \end{aligned}$$

Vectors in M^5 are denoted by underlining.

We represent a fixed orthonormal frame $\underline{e}_A$ for M^5 by the Dirac 4×4 matrices $\underline{\gamma}_A$

$$\underline{\gamma}_0 = i\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \underline{\gamma}_\alpha = \begin{pmatrix} 0 & i\sigma_\alpha \\ -i\sigma_\alpha & 0 \end{pmatrix}, \qquad \underline{\gamma}_4 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

with

$$\underline{\gamma}_A \underline{\gamma}_B + \underline{\gamma}_B \underline{\gamma}_A = 2\eta_{AB} \mathsf{I}$$

where I is the 4×4 unit matrix. $\underline{\gamma}_\alpha$ and $\underline{\gamma}_4$ are Hermitean while $\underline{\gamma}_0$ is anti-Hermitean. We have then for the scalar product of two five-vectors $\underline{y}$ and $\underline{z}$ in M^5

$$\underline{y} \cdot \underline{z} = y^A \eta_{AB} z^B = \mathrm{Tr}\, \underline{y}\underline{z}/4, \qquad y_A = \mathrm{Tr}\, \underline{y}\underline{\gamma}_A/4$$

The Robertson imbedding $\underline{F}\colon \Phi \to M^5$ of the Friedmann manifold Φ into M^5 can be given by

$$\underline{y} = \underline{F}(\tau, X) = 2a\begin{pmatrix} 2i\sin(\tau/2) I & \cos^2(\tau/2) X \\ \cos^2(\tau/2) X^+ & -2i\sin(\tau/2) I \end{pmatrix}$$

The imbedding is isometric since

$$d\underline{y} \cdot d\underline{y} = \mathrm{Tr}\, d\underline{F}\, d\underline{F}/4 = R^2(\tau)(-d\tau^2 + \mathrm{Tr}\, dX\, dX^+/2)$$

We introduce the four four-vectorfields E_j on Φ dual to ω^i by

$$\omega^i(E_j) = \delta^i_j$$

or

$$E_0 = R^{-1}(\tau)\partial/\partial\tau, \qquad E_\beta = R^{-1}(\tau)\varepsilon_\beta, \qquad \lambda^\alpha(\varepsilon_\beta) = \delta^\alpha_\beta$$

The ε_β form the three-leg dual to the one-forms λ^α on $SU(2)$.

The map $\underline{\mathbf{F}}$ of Φ into M^5 induces a map $\underline{\mathbf{F}}_*$ of the vectorfields E_i on Φ into tangent vectors $\underline{\mathbf{E}}_i$ to $\underline{\mathbf{F}}(\Phi)$ in M^5 given by

$$\underline{\mathbf{E}}_i = \underline{\mathbf{F}}_*(E_i) = d\underline{\mathbf{F}}(E_i)$$

or

$$\underline{\mathbf{E}}_0 = (\cos \tau/2)^{-1}\begin{pmatrix} iI & -\sin(\tau/2)X \\ -\sin(\tau/2)X^+ & -iI \end{pmatrix}$$

$$\underline{\mathbf{E}}_\alpha = \begin{pmatrix} 0 & iX\sigma_\alpha \\ -i\sigma_\alpha X^+ & 0 \end{pmatrix}$$

The outer unit normal vector $\underline{\mathbf{E}}_4$ to $\underline{\mathbf{F}}(\Phi)$ is given by

$$\underline{\mathbf{E}}_4 = (\cos \tau/2)^{-1}\begin{pmatrix} -i\sin(\tau/2)I & X \\ X^+ & i\sin(\tau/2)I \end{pmatrix}$$

and we have

$$\underline{\mathbf{E}}_A \cdot \underline{\mathbf{E}}_B = \eta_{AB}$$

The second fundamental form II defined by

$$II = -d\underline{\mathbf{F}} \cdot d\underline{\mathbf{E}}_4 = h_{ij}\omega^i\omega^j$$

becomes

$$II = -(a/2R^3)^{1/2}(\omega^0)^2 - (2a/R^3)^{1/2}\boldsymbol{\omega}\cdot\boldsymbol{\omega} = \chi_i\omega^i$$

4. GLOBAL RIGIDITY

A "motion" π (including reflections) in M^5 is an element of the Poincaré group for M^5 which moves point $\underline{\mathbf{y}}$ into point $\underline{\mathbf{y}}'$

$$\underline{\mathbf{y}}' = \pi(\underline{\mathbf{y}}) = y'^A\underline{\mathbf{e}}_A = (\Lambda^A_B y^B + \mu^A)\underline{\mathbf{e}}_A, \qquad \Lambda^A_B\eta_{AC}\Lambda^C_D = \eta_{BD}$$

It is clear that the imbeddings $\underline{\mathbf{F}}' = \pi\circ\underline{\mathbf{F}}$ obtained by rigidly translating and "rotating" $\underline{\mathbf{F}}(\Phi)$ in M^5 lead also to isometric imbeddings of Φ. These imbeddings comprise, in fact, all the isometric imbeddings of Φ into M^5. We state the following:

Theorem 2 [global rigidity of $\underline{\mathbf{F}}(\Phi)$]. For any isometric imbedding $\underline{\mathbf{F}}'$ of the Friedmann manifold Φ into M^5 there exists a motion π of M^5 into itself such that $\underline{\mathbf{F}}' = \pi\circ\underline{\mathbf{F}}$, where $\underline{\mathbf{F}}$ is the Robertson imbedding. The proof will be given elsewhere (Lauro and Schucking, to be published).

For a given isometric imbedding $\underline{\mathbf{F}}'(\Phi)$ the Λ^A_B can be determined as the solution of the equations

$$E'^A_C(p) = \Lambda^A_B E^B_C(p)$$

for a given point p of Φ. The translation μ^A is then obtained from

$$\mu^A = y'^A(p) - \Lambda^A_B y^B(p)$$

The global rigidity of the imbedding is expressed by the fact that Λ^A_B and μ^A are independent of the choice of the point p of Φ.

5. CONCLUSION

The hypersurface $\underline{F}(\Phi)$ is a submanifold of M^5 and can also be described by the irreducible algebraic equation

$$-4a^2[1-(y^0/4a)^2]^2+(y^1)^2+(y^2)^2+(y^3)^2+(y^4)^2=0, |y^0|<4a$$

which illustrates that $\underline{F}(\Phi)$ can be obtained by "rotating" the segment of the parabolic arc

$$y^4 = 2a[1-(y^0/4a)^2], \qquad |y^0|<4a$$

of the y^0-y^4 plane about the y^0 axis (see Figure 1a).

At the singularities $y^0 = \pm 4a$ the parabolic arc is approximated by its tangents

$$y^4 \pm y^0 = 4a$$

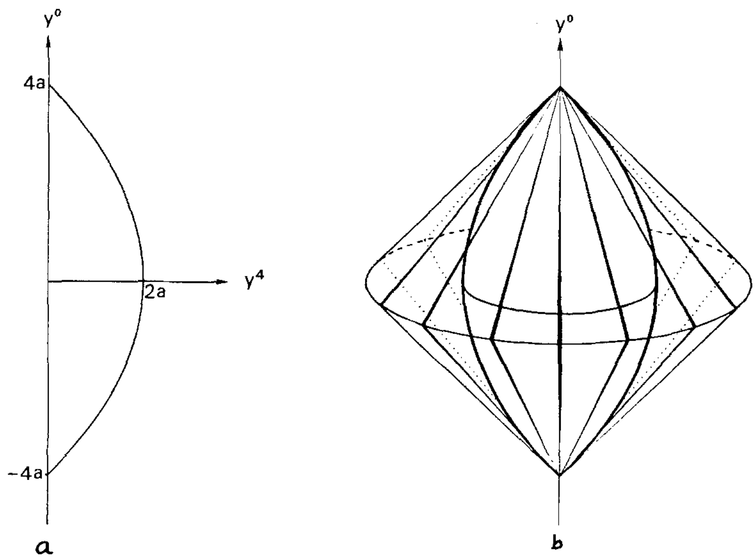

Fig. 1. (a) The arc $y^4 = 2a[1-(y^0/4a)^2]$. (b) $\underline{F}(\Phi)$ with null half-cones.

This leads to an approximation of $\underline{F}(\Phi)$ at the singularities by null half-cones in M^5 (see Figure 1b). This characterization of the singularities holds for a wide class of equations of state which includes, e.g., the radiation cosmos with positive space curvature and vanishing cosmological constant (Schucking and Wang, to be published).

Einstein's theory of gravitation assumes that space-time in the neighborhood of an event can be approximated by a flat Lorentzian manifold tangent to the event. This approximation is the rationale for the belief that local physics at the event is governed by the Poincaré group. It is valid for proper times and lengths small compared to a characteristic length derived from the curvature invariants. Near the big bang the characteristic length and the size of the horizon are of the order ct where t is cosmic time. Poincaré-invariant physics will thus be valid near the big bang only in regions much smaller than ct and will be losing its validity as one approaches the singularity.

Although Einstein's theory breaks down at the singularity it indicates in the highly symmetric case of the Friedmann model the outlines of a kinematic modification. The null-half-cone approximation shows that the manifold Φ approaches near the singularity a singular "manifold" whose connection and degenerate metric are invariant under the ten-dimensional orthochronous de Sitter group down to $t = 0$. The Poincaré group can be obtained from it by group contraction. It would appear, therefore, that fields near the singularity ought to be considered as representations of the de Sitter group. A suitable "regularity" condition for the fields at the vertex of the half-cone might then assure a unity of the universe that remains enigmatic in a Poincaré-invariant picture of a singularity consisting of an infinity of causally disconnected coinciding events.

6. SUMMARY

A new method is proposed for the investigation of the big bang singularity which would appear especially applicable for the post-Planck pre-inflationary era of the very early universe approximated by a radiation cosmos. This procedure of globally rigid imbedding is similar to the familiar method of studying intrinsic properties of the two-sphere S^2—like spherical trigonometry—by using vector algebra of Euclidean three space.

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